Exact conditions for no ruin for the generalised Ornstein-Uhlenbeck process *

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April 10, 2008

Abstract

For a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ the generalised Ornstein-Uhlenbeck (GOU) process is defined as

$$V_t := e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s\right), \quad t \geq 0,$$

where $z \in \mathbb{R}$. We define necessary and sufficient conditions under which the infinite horizon ruin probability for the process is zero. These conditions are stated in terms of the canonical characteristics of the Lévy process and reveal the effect of the dependence relationship between ξ and η. We also present technical results which explain the structure of the lower bound of the GOU.

Keywords: Lévy process, Generalised Ornstein-Uhlenbeck process, Exponential functionals of Lévy processes, Ruin probability

MSC: primary 60H30; secondary 60J25; 91B30

1 Introduction and Notation

For a bivariate Lévy process $(\xi, \eta) = (\xi_t, \eta_t)_{t\geq 0}$ the generalised Ornstein-Uhlenbeck (GOU) process $V = (V_t)_{t\geq 0}$, where $V_0 = z \in \mathbb{R}$, is defined as

$$V_t := e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s\right). \tag{1.1}$$

*The authors wish to dedicate this paper to the memory of Chris Heyde, mentor and friend.

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It is closely related to the stochastic integral process $Z = (Z_t)_{t\geq 0}$ defined as

$$Z_t := \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s. \tag{1.2}$$

The GOU is a time homogenous strong Markov process. For an overview of its properties see Maller et al. [13], and Carmona et al. [4]. Applications are many, and include option pricing (e.g Yor [20]), financial time series (e.g. Klüppelberg et al. [10]), insurance, and risk theory (e.g. Paulsen [17], Nyrhinen [15]).

In this paper, we present some basic foundational results on the ruin probability for the GOU, in a very general setup. There are only a few papers dealing with this, or with passage-time problems for the GOU. Patie [16], and Novikov [14], give first passage-time distributions in the special case that $\xi_t = \lambda t$ for $\lambda \in \mathbb{R}$, and η has no positive jumps. With regard to ruin probability, Nyrhinen [15] and Kalashnikov and Norberg [7] discretize the GOU into a stochastic recurrence equation. Under a variety of conditions, they produce some asymptotic equivalences for the infinite horizon ruin probability. Other work on the GOU ruin probability comes from Paulsen [17]. In the special case that ξ and η are independent, Paulsen gives conditions for certain ruin for the GOU, and a formula for the ruin probability under conditions which ensure that the integral process Z_t converges almost surely as $t \to \infty$.

Since these papers were written, the theory relating to the GOU, and to the process Z, has advanced. In the general case where dependence between ξ and η is allowed, Erickson and Maller [5] present necessary and sufficient conditions for the almost sure convergence of Z_t to a random variable Z_∞ as $t \to \infty$. Bertoin et al. [3] present necessary and sufficient conditions for continuity of the distribution of Z_∞ given it exists. Lindner and Maller [11] show that strict stationarity of V is equivalent to convergence of an integral $\int_0^t e^{\xi_{s-}} \mathrm{d}L_s$, where L is an auxiliary Lévy process composed of elements of ξ and η. Note that in [11] the sign of the process ξ is reversed in the definition of the GOU. For our purposes it suits to have the GOU in the form $V_t := e^{\xi_t}(z + Z_t)$ and to study the behaviour of V in terms of Z.

Our main results are presented in Section 2. Theorem 2.1 presents exact necessary and sufficient conditions under which the infinite horizon ruin probability for the GOU is zero. These conditions do not relate to the convergence of Z or stationarity of V or to any moment conditions. Instead they are are expressed at a more basic level, directly on the Lévy measure of (ξ, η). Theorem 2.3 shows that $P(Z_t < 0) > 0$ for all $t > 0$ as long as η is not a subordinator. This result is an important building block in the proof of Theorem 2.1. Finally in Section 2, Theorem 2.4 extends a ruin probability formula in Paulsen [17], presenting a slightly different version which deals with the general dependent case, and applies whenever Z_t converges almost surely to a random variable Z_∞ as $t \to \infty$.

Section 3 contains technical results of interest, which characterise what we call the lower bound function of the GOU, and are used to prove the main ruin probability theorem. Section 4 contains proofs of the results stated in Sections 2 and 3.

1.1 Notation

We now set out our theoretical framework and notation. Let $(X_t)_{t\geq 0} := (\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process with $\xi_0 = \eta_0 = 0$, adapted to a filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{0\leq t\leq\infty}, P)$ satisfying the "usual hypotheses" (see Protter [18] p.3), where ξ and η are not identically zero. Assume the σ-algebra $\mathscr{F}$ and the filtration $\mathbb{F}$ are generated by (ξ, η), that is, $\mathscr{F} := \sigma\left((\xi,\eta)_t : 0 \leq t < \infty\right)$ and $\mathscr{F}_t := \sigma\left((\xi,\eta)_s : 0 \leq s \leq t\right)$. Note that the processes V and Z are defined with respect to $\mathbb{F}$.

The characteristic triplet of (ξ, η) will be written $((\tilde{\gamma}_\xi, \tilde{\gamma}_\eta), \Sigma_{\xi,\eta}, \Pi_{\xi,\eta})$ where $(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta) \in \mathbb{R}^2$, the Gaussian covariance matrix $\Sigma_{\xi,\eta}$ is a non-stochastic 2×2 positive definite matrix, and the Lévy measure $\Pi_{\xi,\eta}$ is a $\sigma-$finite measure on $\mathbb{R}^2 \setminus \{0\}$ satisfying the condition $\int_{\mathbb{R}^2} \min\{|z|^2, 1\}\Pi_{\xi,\eta}(\mathrm{d}z) < \infty$, where $|\cdot|$ denotes Euclidean distance. For details on Lévy processes see Bertoin [2] and Sato [19].

The Lévy-Ito decomposition (Sato [19], Ch.4,) breaks down (ξ, η) into a sum of four mutually independent Lévy processes:

$$
\begin{aligned}
(\xi_t, \eta_t) \quad &= \quad (\tilde{\gamma}_\xi, \tilde{\gamma}_\eta)t + (B_{\xi,t}, B_{\eta,t}) + \int_{|z|<1} z\left(N_{\xi,\eta,t}(\cdot, \mathrm{d}z) - t\Pi_{\xi,\eta}(\mathrm{d}z)\right) \\
&\quad + \int_{|z|\geq 1} z N_{\xi,\eta,t}(\cdot, \mathrm{d}z), \qquad (1.3)
\end{aligned}
$$

where B_ξ and B_η are Brownian motions such that (B_ξ, B_η) has covariance matrix $\Sigma_{\xi,\eta}$, and $N_{\xi,\eta,t}(\omega,\ )$ is the random jump measure of (ξ,η) such that $E\left(N_{\xi,\eta,1}(\omega,\Lambda)\right) = \Pi_{\xi,\eta}(\Lambda)$ for Λ a Borel subset of $\mathbb{R}^2 \setminus \{0\}$ whose closure does not contain 0. We can write (see Protter [18], p.31)

$$
(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta) = E\left((\xi_1, \eta_1) - \int_{|z|\geq 1} z N_{\xi,\eta,1}(\cdot, \mathrm{d}z)\right). \qquad (1.4)
$$

The characteristic triplets of ξ and η as one-dimensional Lévy processes are denoted $(\gamma_\xi, \sigma_\xi^2, \Pi_\xi)$ and $(\gamma_\eta, \sigma_\eta^2, \Pi_\eta)$ respectively, where

$$
\Pi_\xi(\Gamma) = \Pi_{\xi,\eta}(\Gamma \times \mathbb{R}) \ \text{ and } \ \Pi_\xi(\Gamma) = \Pi_{\xi,\eta}(\mathbb{R} \times \Gamma) \qquad (1.5)
$$

for Γ a Borel subset of $\mathbb{R} \setminus \{0\}$ whose closure does not contain 0,

$$
(\gamma_\xi, \gamma_\eta) = (\tilde{\gamma}_\xi, \tilde{\gamma}_\eta) + \int_{\{|x|\leq 1, |y|>\sqrt{1-x^2}\}} (x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)), \qquad (1.6)
$$

and σ_ξ^2 and σ_η^2 are the upper left and lower right entries respectively, in the matrix $\Sigma_{\xi,\eta}$. Analagous to (1.3), we can write the Lévy-Ito decomposition of ξ as

$$\xi_t = \gamma_\xi t + B_{\xi,t} + \int_{|x|<1} x\left(N_{\xi,t}(\cdot,\mathrm{d}x) - t\Pi_\xi(\mathrm{d}x)\right) + \int_{|x|\geq 1} xN_{\xi,t}(\cdot,\mathrm{d}x), \tag{1.7}$$

where

$$\gamma_\xi = E\left(\xi_1 - \int_{|x|\geq 1} xN_{\xi,1}(\cdot,\mathrm{d}x)\right), \tag{1.8}$$

and similarly for η. For further details on Lévy-Ito decompositions, see Sato [19], Chapter 4.

A Lévy process is said to be a subordinator if it takes only non-negative values, which implies that its sample paths are non-decreasing (Bertoin [2], p.71).

Stochastic integrals are interpreted according to Protter [18]. The integral $\int_a^b$ is interpreted as $\int_{[a,b]}$ and the integral $\int_{a+}^b$ as $\int_{(a,b]}$. The jump of a process Y at t is denoted by $\Delta Y_t := Y_t - Y_{t-}$. The Lévy measure of a Lévy process Y is denoted by Π_Y. If T is a fixed time or a stopping time denote the process Y stopped at T by Y^T and define it by $Y_t^T := Y_{t\wedge T} := Y_{\min\{t,T\}}$. For a function $f(x)$ define $f^+(x) := f(x) \vee 0 := \max\{f(x), 0\}$ and $f^-(x) := \max\{-f(x), 0\}$. The symbol 1_Λ will denote the characteristic function of a set Λ. The symbol $=_D$ will denote equality in distribution of two random variables. The initials "iff" will denote the phrase "if and only if". The symbol "a.s" will denote equality, or convergence, almost surely. Let T_z denote the first time V drops below zero, so

$$T_z := \inf\left\{t > 0 : V_t < 0 \middle| V_0 = z\right\}$$

and $T_z := \infty$ whenever $V_t > 0 \ \ \forall t > 0$ and $V_0 = z$. For $z \geq 0$, define the infinite horizon ruin probability function to be

$$\psi(z) := P\left(\inf_{t\geq 0} V_t < 0 \middle| V_0 = z\right) = P(T_z < \infty).$$

2 Ruin Probability Results

Our results are given in terms of regions of support of the Lévy measure $\Pi_{\xi,\eta}$. We define some notation, beginning with the following quadrants of the plane. Let $A_1 := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, and similarly, let A_2, A_3 and A_4 be the quadrants in which $\{x \geq 0, y \leq 0\}$, $\{x \leq 0, y \leq 0\}$ and $\{x \leq 0, y \geq 0\}$ respectively. For each $i = 1, 2, 3, 4$ and $u \in \mathbb{R}$ define

$$A_i^u := \left\{(x,y) \in A_i : y - u(e^{-x} - 1) < 0\right\}.$$

These sets are defined such that if $(\Delta\xi_t, \Delta\eta_t) \in A_i^u$ and $V_{t-} = u$, then $\Delta V_t < 0$, as we see from the equation

$$\begin{aligned}
\Delta V_t &= V_t - V_{t-} \\
&= e^{\xi_t}\Big(z + \int_0^{t-} e^{-\xi_{s-}}\mathrm{d}\eta_s + e^{-\xi_t}\Delta\eta_t\Big) - e^{\xi_{t-}}\Big(z + \int_0^{t-} e^{-\xi_{s-}}\mathrm{d}\eta_s\Big) \\
&= (e^{\xi_t} - e^{\xi_{t-}})\Big(z + \int_0^{t-} e^{-\xi_{s-}}\mathrm{d}\eta_s\Big) + e^{\xi_t}e^{\xi_{t-}}\Delta\eta_t \\
&= (e^{\Delta\xi_t} - 1)V_{t-} + e^{\Delta\xi_t}\Delta\eta_t.
\end{aligned} \tag{2.9}$$

If $u \le 0$ then $A_2^u = A_2$ and $A_4^u = \emptyset$. As u decreases to $-\infty$, the sets A_1^u shrink, whilst A_3^u expand. Define

$$\theta_1 := \begin{cases} \sup\{u \le 0 : \Pi_{\xi,\eta}(A_1^u) > 0\} \\ -\infty \quad \text{if } \Pi_{\xi,\eta}(A_1) = 0 \end{cases}, \theta_3 := \begin{cases} \inf\{u \le 0 : \Pi_{\xi,\eta}(A_3^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_3) = 0 \end{cases}.$$

If $u \ge 0$ then $A_3^u = A_3$ and $A_1^u = \emptyset$. As u increases to ∞, the sets A_2^u shrink, whilst A_4^u expand. Define

$$\theta_2 := \begin{cases} \sup\{u \ge 0 : \Pi_{\xi,\eta}(A_2^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_2) = 0, \end{cases}, \theta_4 := \begin{cases} \inf\{u \ge 0 : \Pi_{\xi,\eta}(A_4^u) > 0\} \\ \infty \quad \text{if } \Pi_{\xi,\eta}(A_4) = 0 \end{cases}.$$

For each $i = 1, 2, 3, 4$, note that $\Pi_{\xi,\eta}(A_i^{\theta_i}) = 0$, since in the definitions of A_i^u we are requiring that $y - u(e^{-x} - 1)$ be strictly less than zero.

Theorem 2.1 (Exact conditions for no ruin for the GOU)**.** *The ruin probability $\psi(z) = 0$ for large enough $z \ge 0$ if and only if the Lévy measure satisfies $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \le \theta_4$, and:*

- *when $\sigma_\xi^2 \ne 0$ the Gaussian covariance matrix is of form $\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2$ for some $u \in [\theta_2, \theta_4]$ satisfying*

$$\tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - \frac{1}{2}u\sigma_\xi^2 - \int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}} (ux + y)\Pi_{\xi,\eta}(\mathrm{d}(x, y)) \ge 0; \tag{2.10}$$

- *when $\sigma_\xi^2 = 0$ the Gaussian covariance matrix is of form $\Sigma_{\xi,\eta} = 0$ and there exists $u \in [\theta_2, \theta_4]$ satisfying (2.10).*

If $\sigma_\xi^2 \ne 0$ and the conditions of the theorem hold, then $\psi(z) = 0$ for all $z \ge u := \frac{\sigma_\eta}{\sigma_\xi}$, whilst $\psi(z) > 0$ for all $z < u$.

If $\sigma_\xi^2 = 0$ and the conditions of the theorem hold, then $\psi(z) = 0$ for all $z \ge u' :=$ $\max\{\theta_2, \inf\{u > 0 : (2.10) \text{ holds}\}\}$, whilst $\psi(z) > 0$ for all $z < u'$.

We now discuss some examples and special cases which illustrate and amplify the results in Theorem 2.1.

Remark 2.2. (*a*) Suppose that (ξ, η) is continuous. We can then write $(\xi_t, \eta_t) = (\gamma_\xi t, \gamma_\eta t) + (B_{\xi,t}, B_{\eta,t})$. Theorem 2.1 states that $\psi(z) = 0$ for all $z \geq u$ and $\psi(z) > 0$ for all $z < u$, if and only if there exists $u > 0$ such that $B_\eta = -uB_\xi$, and $(\gamma_\xi - \frac{1}{2}\sigma_\xi^2)u + \gamma_\eta \geq 0$. For example we could have

$$(\xi_t, \eta_t) := (B_t + ct, -B_t + (1/2 - c)t), \tag{2.11}$$

where $c \in \mathbb{R}$. Then Theorem 2.1 implies that $\psi(z) = 0$ for all $z \geq u = \frac{\sigma_\eta}{\sigma_\xi} = 1$ whilst $\psi(z) > 0$ for all $z < 1$. In this simple case, we can check the result directly. Using Ito's formula we obtain

$$Z_t = -\int_0^t e^{-(B_s+cs)} \mathrm{d}B_s + (1/2 - c)\int_0^t e^{-(B_s+cs)} \mathrm{d}s = e^{-(B_s+cs)} - 1,$$

and hence a lower bound for Z is -1.

(*b*) Suppose that (ξ, η) is a finite variation Lévy process. Then we must have $\Sigma_{\xi,\eta} = 0$ and $\int_{|z|<1} |z|\Pi_{\xi,\eta}(\mathrm{d}z) < \infty$. We can define the drift vector as

$$(d_\xi, d_\eta) := \gamma_\eta - \int_{|z|<1} z\Pi_{\xi,\eta}(\mathrm{d}z)$$

and write

$$(\xi_t, \eta_t) = (d_\xi, d_\eta)t + \int_{\mathbb{R}^2} zN_{\xi,\eta,t}(\cdot, \mathrm{d}z).$$

In this situation, the conditions of Theorem 2.1 can be made more explicit. Theorem 2.1 states that $\psi(z) = 0$ for large enough z if and only if $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \leq \theta_4$, and at least one of the following is true:

- $d_\xi = 0$, and $d_\eta \geq 0$; or
- $d_\xi > 0$ and $-\frac{d_\eta}{d_\xi} \leq \theta_4$; or
- $d_\eta > 0$, and $d_\xi < 0$, such that $-\frac{d_\eta}{d_\xi} \geq \theta_2$.

If the second property holds, then $\psi(z) = 0$ for all $z \geq \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$ and $\psi(z) > 0$ for all $z < \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$. If the other properties hold, then $\psi(z) = 0$ for all $z \geq \theta_2$ and $\psi(z) > 0$ for all $z < \theta_2$.

These results follow easily by transforming condition (2.10) into conditions on (d_ξ, d_η). For a simple example, let N_t be a Poisson process with parameter λ, let $c > 0$ and let

$$(\xi_t, \eta_t) := (-ct + N_t, 2ct - N_t). \tag{2.12}$$

Then we are in the third case above, and $\psi(z) = 0$ for all $z \geq \theta_2 = \frac{e}{e-1}$, and $\psi(z) > 0$ for all $z < \frac{e}{e-1}$. In this simple case, we can verify the results by direct but tedious calculations which we omit here.

(*c*) The case in which ξ and η are independent is analysed in Paulsen [17]. In the cases $E(\xi_1) < 0$ and $E(\xi_1) = 0$, and under certain moment conditions, he shows that $\psi(z) = 1$ for all $z \geq 0$. Theorem 2.1 shows that the situation changes when dependence is allowed. The continuous process defined in (2.11), and the jump process defined in (2.12), illustrate this difference. Each process trivially satisfies Paulsen's moment conditions and can satisfy $E(\xi_1) < 0$, or $E(\xi_1) = 0$, depending on the choices of c and λ, however it is not the case that $\psi(z) = 1$ for all $z \geq 0$.

(*d*) If η is a subordinator then $Z_t \geq 0$ for all $t \geq 0$, and hence $\psi(z) = 0$ for all $z \geq 0$. Theorem 2.1 agrees with this trivial case. By Sato [19], p.137, η is a subordinator if and only if the following three conditions hold:

- $\sigma_\eta^2 = 0$, so η has no Brownian component;
- $\Pi_\eta((-\infty, 0)) = 0$, so η has no negative jumps;
- $d_\eta \geq 0$, where

$$d_\eta := \gamma_\eta - \int_{(0,1)} y \Pi_\eta(\mathrm{d}y) = E\left(\eta_1 - \int_{(0,\infty)} y N_{\eta,1}(\cdot, \mathrm{d}y)\right).$$

Note that when $\Pi_\eta((-\infty, 0)) = 0$,then d_η exists and $d_\eta \in [-\infty, \infty)$, where $d_\eta = -\infty$ iff $\int_{(0,1)} y \Pi_\eta(\mathrm{d}y) = \infty$.

Now $\sigma_\eta^2 = 0$ implies that $\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sigma_\xi^2$. When η has no negative jumps, then $\Pi_{\xi,\eta}(A_3) = 0 = \Pi_{\xi,\eta}(A_2)$, and hence $0 = \theta_2 \leq \theta_4$. The third property, $d_\eta \geq 0$, implies that (2.10) is satisfied for $u = 0$, since (1.6) implies that

$$\begin{aligned} \tilde{\gamma}_\eta - \int_{\{y>0\}\cap\{x^2+y^2<1\}} y \Pi_{\xi,\eta}(\mathrm{d}(x,y)) &= \gamma_\eta - \int_{(-1,1)\times(0,1)} y \Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\ &= d_\eta. \end{aligned}$$

Hence, Theorem 2.1 verifies that $\psi(z) = 0$ for all $z \geq u = 0$.

(*e*) The expression on the left hand side of (2.10) always exists whenever the remaining conditions of the theorem are satisfied, however it may have the value $-\infty$. If all conditions of the theorem are satisfied then

$$\int_{\{y-u(e^{-x}-1)\in(0,1)\}} \left(y - u(e^{-x} - 1)\right) \Pi_{\xi,\eta}(\mathrm{d}(x,y)) < \infty. \tag{2.13}$$

On first viewing, (2.13) may seem counterintuitive, as it places a constraint on the size of the positive jumps of V. However, if (2.13) does not hold, and all the other conditions, excluding (2.10), are satisfied, then the Lévy properties of (ξ, η) imply that V_t can drift negatively when $V_{t-} = u$. These statements are discussed further in Remark 3.5 following Theorem 3.4.

Theorem 2.3. *The Lévy process η is not a subordinator if and only if $P(Z_T < 0) > 0$ for any fixed time $T > 0$.*

One direction of this result is trivial and has been noted above, namely, if η is a subordinator then $P(Z_T < 0) = 0$ for any $T > 0$. The other direction seems quite intuitive and in fact is implicitly assumed by Paulsen [17] in the case when ξ and η are independent. However even in the independent case the proof is non-trivial. We prove it in the general case using a change of measure argument and some analytic lemmas. As well as being of independent interest, this result is essential in proving Theorem 2.1.

The final theorem in this section provides a formula for the ruin probability in the case that Z converges. Recall that T_z denotes the first time V drops below zero when $V_0 = z$, or equivalently, the first time Z drops below $-z$.

Theorem 2.4. *Suppose Z_t converges a.s to a finite random variable Z_∞ as $t \to \infty$, and let $G(z) := P(Z_\infty \le z)$. Then*

$$\psi(z) = \frac{G(-z)}{E\left(G(-V_{T_z})\middle|T_z < \infty\right)}.$$

Note that $G(-V_{T_z})(\omega) := P\left(\nu \in \Omega \ : \ Z_\infty(\nu) < -V_{T_z}(\omega)\right)$. It is defined whenever $T_z(\omega) < \infty$.

Remark 2.5. (*a*) In the case that ξ and η are independent, Paulsen [17] shows, under a number of side conditions which ensure that Z_t converges a.s to a finite random variable Z_∞ with distribution function $H(z) := P(Z_\infty < z)$ as $t \to \infty$, that

$$\psi(z) = \frac{H(-z)}{E\left(H(-V_{T_z})\middle|T_z < \infty\right)}.$$

This formula is a modification of a result given by Harrison [6] for the special case in which ξ is deterministic drift and η is a Lévy process with finite variance. Theorem 2.4 extends the formula to the general dependent case. Our proof is similar to those of Paulsen and Harrison, however we write it out in full because some details are different.

(b) Erickson and Maller [5] prove that Z_t converges a.s to a finite random variable Z_∞ as $t \to \infty$ if and only if

$$\lim_{t\to\infty} \xi_t = +\infty \ a.s \quad \text{and} \quad \int_{\mathbb{R}\setminus[-e,e]} \left(\frac{\ln |y|}{A_\xi(\ln |y|)} \right) \Pi_\eta(\mathrm{d}y) < \infty,$$

where, for $x \geq 1$,

$$A_\xi(x) := 1 + \int_1^x \Pi_\xi((z,\infty))\mathrm{d}z.$$

Lindner and Maller [11] prove that if V is not a constant process, then V is strictly stationary if and only if $\int_0^\infty e^{\xi_{s-}}\mathrm{d}L_s$ converges a.s to a finite random variable as $t \to \infty$, where L is the Lévy process

$$L_t := \eta_t + \sum_{0<s\leq t} \left(e^{-\Delta\xi_s} - 1 \right) \Delta\eta_s - t\mathrm{Cov}(B_{\xi,1}, B_{\eta,1}), \quad t \geq 0.$$

In neither of these cases do the conditions of Theorem 2.1 simplify. Each of the processes defined in (2.11) and (2.12) can belong to either of these cases, or neither, depending on the choice of constant c and parameter λ.

(c) Bertoin et al. [3] prove that if Z_t converges a.s to a finite random variable Z_∞ as $t \to \infty$, then Z_∞ has an atom iff Z_∞ is a constant value k iff $P\left(Z_t = k(1 - e^{-\xi_t}) \ \forall t > 0\right) = 1$ iff $e^{-\xi} = \epsilon(-\eta/k)$, where $\epsilon(\cdot)$ denotes the stochastic exponential. In this case it is trivial that $\psi(z) = 0$ for all $z \geq -k$. Theorem 2.1 produces the same result, however this will not become immediately clear until Remark 3.3 (2) following Theorem 3.2.

3 Technical Results of Interest

This section contains technical results needed in the proofs of Theorems 2.1 and 2.3, which also have some independent interest. Recall that the stochastic, or Doléans-Dade, exponential of a semimartingale W_t is denoted by $\epsilon(W)_t$.

Proposition 3.1. *Given a Lévy process ξ with characteristic triplet $(\gamma_\xi, \sigma_\xi, \Pi_\xi)$ there exists a Lévy process W adapted to the same filtration, such that $e^{-\xi_t} = \epsilon(W)_t$, where (ξ, W) is the bivariate Lévy process with characteristic triplet $((\tilde{\gamma}_\xi, \tilde{\gamma}_W), \Sigma_{\xi,W}, \Pi_{\xi,W})$ defined as follows:*

$$\Sigma_{\xi,W} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sigma_\xi^2, \tag{3.14}$$

the Lévy measure $\Pi_{\xi,W}$ is concentrated on $\{(x, e^{-x} - 1) \ : \ x \in \mathbb{R}\}$ so that

$$\Pi_W((-\infty, -1]) = 0$$

and

$$\Pi_W(\Lambda) = \Pi_\xi(-\ln(\Lambda+1)) \qquad \textit{when } \Lambda \subset (-1,\infty),$$

and

$$\tilde{\gamma}_\xi + \tilde{\gamma}_W = \frac{1}{2}\sigma_\xi^2 + \int_{x^2+(e^{-x}-1)^2<1} (x + e^{-x} - 1)\Pi_\xi(\mathrm{d}x). \tag{3.15}$$

We define the lower bound function δ for V in (1.1) as

$$\delta(z) = \inf\left\{u \in \mathbb{R} : P\left(\inf_{t\geq 0} V_t \leq u \middle| V_0 = z\right) > 0\right\}.$$

The following theorem exactly characterizes the lower bound function.

Theorem 3.2. *The lower bound function satisfies the following properties:*

(*a*) *For all $z \in \mathbb{R}$, $\delta(z) \leq z$.*

(*b*) *If $z_1 < z_2$ then $\delta(z_1) \leq \delta(z_2)$.*

(*c*) *Let W be the Lévy process such that $e^{-\xi_t} = \epsilon(W)_t$. Then $\delta(z) = z$ if and only if $\eta - zW$ is a subordinator.*

(*d*) *For all $z \in \mathbb{R}$, $\delta(z) = \delta(\delta(z))$, and*

$$\delta(z) = \sup\left\{u : u \leq z, \eta - uW \textit{ is a subordinator}\right\}.$$

Remark 3.3. (*a*) If η is a subordinator then $\delta(0) = 0$, so V cannot drop below zero when $V_0 = z \geq 0$.

(*b*) As noted in Remark 2.5 (3), if Z_t converges a.s to a finite random variable Z_∞ as $t \to \infty$, then Z_∞ has an atom iff $e^{-\xi} = \epsilon(-\eta/k)$. If this holds then $\delta(-k) = -k$, since $\eta + k(-\eta/k) = 0$ and hence is a subordinator. Thus $\psi(z) = 0$ for all $z \geq -k$, as mentioned in Remark 2.5 (3).

Theorem 3.4. *Let $u \in \mathbb{R}$. With W defined as in Proposition 3.1, the Lévy process $\eta - uW$ is a subordinator if and only if the following three conditions are satisfied: the Gaussian covariance matrix is of the form*

$$\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2, \tag{3.16}$$

at least one of the following is true:

- $\Pi_{\xi,\eta}(A_3) = 0$ *and* $\theta_2 \leq \theta_4$ *and* $u \in [\theta_2, \theta_4]$;

- $\Pi_{\xi,\eta}(A_2) = 0$ *and* $\theta_1 \leq \theta_3$ *and* $u \in [\theta_1, \theta_3]$;
- $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ *and* $u \in [\theta_1, \theta_4]$;

and in addition, u satisfies (2.10).

Remark 3.5. In Remark 2.2 (4) we stated three necessary and sufficient conditions for a Lévy process to be a subordinator. These three conditions correspond respectively with the three conditions in Theorem 3.4, as we shall see in the proof. In particular, (2.10) is equivalent to the condition $d_{\eta-uW} \geq 0$. As noted in Remark 2.2 (4), if the first two conditions of Theorem 3.4 hold, then $d_{\eta-uW} \in [-\infty, \infty)$, thus ensuring that (2.10) is well defined. Further, if all three conditions hold, then $\int_{(0,1)} z\Pi_{\eta-uW}(\mathrm{d}z) < \infty$, which we will show to be equivalent to (2.13). Note that if $\eta - uW$ has no Brownian component, no negative jumps, but $\int_{(0,1)} z\Pi_{\eta-uW}(\mathrm{d}z) = \infty$, then, somewhat suprisingly, $\eta - uW$ is fluctuating and hence not a subordinator, regardless of the value of the shift constant $\gamma_{\eta-uW}$. This behaviour occurs since $d_{\eta-uW} = -\infty$, and is explained in Sato [19], p138.

4 Proofs

We begin by proving Theorem 2.3. For this proof, some lemmas are required. In these we assume that $X = (\xi, \eta)$ has bounded jumps so that X has finite absolute moments of all orders. Then, to prove Theorem 2.3 we reduce to this case.

Lemma 4.1. *Suppose* $X = (\xi, \eta)$ *has bounded jumps and* $E(\eta_1) = 0$. *If we let* $T > 0$ *be a fixed time then* Z^T *is a mean-zero martingale with respect to* $\mathbb{F}$.

Proof. Since η is a Lévy process the assumption $E(\eta_1) = 0$ implies that η is a càdlàg martingale. Since ξ is càdlàg , $e^{-\xi}$ is a locally bounded process and hence Z is a local martingale for $\mathbb{F}$ by Protter [18], p.171. If we show that $E\left(\sup_{s\leq t}|Z_s^T|\right) < \infty$ for every $t \geq 0$ then Protter [18], p.38 implies that Z^T is a martingale. This is equivalent to showing $E\left(\sup_{t\leq T}|Z_t|\right) < \infty$. Since Z is a local martingale and $Z_0 = 0$, the Burkholder-Davis-Gundy inequalities in Lipster and Shiryaev [12], p.70 and p.75, ensure the existence of

$b > 0$ such that

$$\begin{aligned} E\left(\sup_{0\le t\le T}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s\right|\right) &\le bE\left(\left[\int_0^\bullet e^{-\xi_{s-}}\mathrm{d}\eta_s, \int_0^\bullet e^{-\xi_{s-}}\mathrm{d}\eta_s\right]_T^{1/2}\right) \\ &= bE\left(\left(\int_0^T e^{-2\xi_{s-}}\mathrm{d}[\eta,\eta]_s\right)^{1/2}\right) \\ &\le bE\left(\left(\int_0^T \sup_{0\le t\le T} e^{-2\xi_t}\mathrm{d}[\eta,\eta]_s\right)^{1/2}\right) \\ &= bE\left(\sup_{0\le t\le T} e^{-\xi_t}[\eta,\eta]_T^{1/2}\right) \\ &\le b\left(E\left(\sup_{0\le t\le T} e^{-2\xi_t}\right)\right)^{1/2}\left(E\left([\eta,\eta]_T\right)\right)^{1/2}, \end{aligned}$$

where the second inequality follows from the fact that $[\eta,\eta]_s$ is increasing and the final inequality follows by the Cauchy-Schwarz inequality. (The notation $[\cdot,\cdot]$ denotes the quadratic variation process.) Now

$$E\left([\eta,\eta]_T\right) = \sigma_\eta^2 T + E\left(\sum_{0\le s\le T}(\Delta\eta)^2\right) = \sigma_\eta^2 T + T\int x^2\Pi_\eta(\mathrm{d}x),$$

which is finite since η has bounded jumps. Thus it suffices to prove $E\left(\sup_{0\le t\le T} e^{-2\xi_t}\right) < \infty$. Setting $Y_t = e^{-\xi_t}/E(e^{-\xi_t})$, a non-negative martingale, it follows by Doob's maximal inequality, as expressed in Shiryaev [1], p.765, that

$$E\left(\sup_{0\le t\le T}\frac{e^{-2\xi_t}}{\left(E(e^{-\xi_t})\right)^2}\right) \le 4\frac{E\left(e^{-2\xi_T}\right)}{\left(E(e^{-\xi_T})\right)^2},$$

which is finite since ξ has bounded jumps and hence has finite exponential moments of all orders (Sato [19], p.161). It is shown in Sato [19], p.165, that $\left(E(e^{-\xi_t})\right)^2 = \left(E(e^{-\xi_1})\right)^{2t}$. Letting $c := \left(E(e^{-\xi_1})\right)^2 \in (0,\infty)$, the above inequality implies that

$$E\left(\sup_{0\le t\le T} e^{-2\xi_t}\right) \le \max\{1, c^T\}E\left(\sup_{0\le t\le T}\frac{e^{-2\xi_t}}{c^t}\right) < \infty.$$

□

We now present two lemmas dealing with absolute continuity of measures. These lemmas will be used to construct a new process W such that W^T is a mean-zero martingale which is mutually absolutely continuous with Z^T. Then $P(Z_T < 0) > 0$ if and only if $P(W_T < 0) > 0$, and the latter statement will follow immediately from the fact that W^T is a mean-zero martingale.

Lemma 4.2. *Let $X := (\xi, \eta)$ and $Y := (\tau, \nu)$ be bivariate Lévy processes adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, and let $Z_t := \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s$ and $W_t := \int_0^t e^{-\tau_{s-}} \mathrm{d}\nu_s$. If the induced probability measures of X^T and Y^T are mutually absolutely continuous, then the induced probability measures of Z^T and W^T are mutually absolutely continuous.*

Proof. Let $D([0,T] \to \mathbb{R}^2)$ denote the set of càdlàg functions from $[0,T]$ to $\mathbb{R}^2$ and $\mathscr{B}^{2[0,T]}$ denote the σ-algebra generated in this set by the Borel cylinder sets (see Kallenberg [8]). Then the induced probability measures of X^T and Y^T can be written as P_{X^T} and P_{Y^T} on the measure space $\big(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}\big)$. Let $C := (C', C'')$ be the coordinate mapping of $\big(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}\big)$ to itself. Define the process Z' on the probability space $\big(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}, P_{X^T}\big)$ by $Z'_t := \int_0^t e^{-C'_{s-}} \mathrm{d}C''_s$. Define W' on $\big(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}, P_{Y^T}\big)$ by $W'_t := \int_0^t e^{-C'_{s-}} \mathrm{d}C''_s$. Note that Z' and W' are different processes since they are being evaluated under different measures. Now $Z = X \circ Z'$ and $W = Y \circ W'$. Hence $P(Z^T \in \Lambda) = P_{X^T}(Z' \in \Lambda)$ and $P(W^T \in \Lambda) = P_{Y^T}(W' \in \Lambda)$. Since P_{X^T} and P_{Y^T} are mutually absolutely continuous, Protter [18], p.60 implies that Z' and W' are P_{X^T}-indistinguishable, and P_{Y^T}-indistinguishable. So $P_{X^T}(Z' \in \Lambda) = P_{X^T}(W' \in \Lambda)$. Since P_{X^T} and P_{Y^T} are mutually absolutely continuous $P_{X^T}(W' \in \Lambda) = 0$ iff $P_{Y^T}(W' \in \Lambda) = 0$ which proves $P(Z^T \in \Lambda) = 0$ iff $P(W^T \in \Lambda) = 0$, as required. □

Lemma 4.3. *If $X := (\xi, \eta)$ has bounded jumps, $E(\eta_1) \geq 0$, η is not a subordinator, and η is not pure deterministic drift, then there exists a bivariate Lévy process $Y := (\tau, \nu)$ with bounded jumps, adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, such that X^T and Y^T are mutually absolutely continuous for all $T > 0$, and $E(\nu_1) = 0$.*

Proof. As mentioned in Remark 2.2 (4,) the Lévy process η is a subordinator if and only if the following three conditions hold: $\sigma_\eta^2 = 0$, $\Pi_\eta((-\infty, 0)) = 0$, and $d_\eta \geq 0$ where $d_\eta := \gamma_\eta - \int_{(0,1)} y \Pi_\eta(\mathrm{d}y)$. Thus it suffices to prove the lemma in the following three cases.

Case 1: Suppose $\sigma_\eta \neq 0$. Given dependent Brownian motions B_ξ and B_η there exists a Brownian motion B' independent of B_η, and constants a_1 and a_2 such that $(B_\xi, B_\eta) = (a_1 B' + a_2 B_\eta\ ,\ B_\eta)$. Using the Lévy-Ito decomposition, X can be written as the sum of two independent processes as follows;

$$\begin{aligned} X_t = (\xi_t\ ,\ \eta_t) &= (\xi'_t + B_{\xi,t}\ ,\ \eta'_t + B_{\eta,t}) \\ &=_D (\xi'_t + a_1 B'_t\ ,\ \eta'_t) + (a_2 B_{\eta,t}\ ,\ B_{\eta,t}), \end{aligned}$$

where $(\xi'\ ,\ \eta')$ is a pure jump Lévy process with drift, independent of $(B_\xi\ ,\ B_\eta)$. Let $c := E(\eta_1)$ and define the Lévy process Y by

$$Y_t := (\xi'_t + a_1 B'_t\ ,\ \eta'_t) + (a_2(B_{\eta,t} - ct)\ ,\ B_{\eta,t} - ct)\,.$$

It is a simple consequence of Girsanov's theorem for Brownian motion, e.g. Klebaner [9], p.241, that the induced measures of $B_{\eta,t}$ and $B_{\eta,t} - ct$ on $\big(D([0,T] \to \mathbb{R}), \mathscr{B}^{[0,T]}\big)$ are

mutually absolutely continuous. It is trivial to show that this implies that the induced probability measures of $(a_2 B_{\eta,t}\ ,\ B_{\eta,t})^T$ and $(a_2(B_{\eta,t}-ct)\ ,\ B_{\eta,t}-ct)^T$ are mutually absolutely continuous. Using independence, this implies that the induced probability measures of X^T and Y^T are mutually absolutely continuous. Note that if we write Y as $Y=(\tau,\nu)$ then $\nu_t=\eta_t-ct$ so $E(\nu_1)=0$ as required.

Case 2: Suppose $\sigma_\eta=0$ and $\Pi_\eta((-\infty,0))>0$. We can assume that X has jumps contained in Λ, a square in $\mathbb{R}^2$, i.e for all $t>0$

$$(\Delta\xi_t,\Delta\eta_t)\in\Lambda:=\{(x,y)\in\mathbb{R}^2:-a\le x\le a,-a\le y\le a\}.$$

For any $0<b<a$ define the set $\Gamma\subset\Lambda$ by

$$\Gamma:=\{(x,y)\in\mathbb{R}^2:-a\le x\le a,-a\le y\le -b\}.$$

A Lévy measure is σ-finite and $\Pi_\eta((-\infty,0))>0$ so there must exist a $b>0$ small enough such that $\Pi_X(\Gamma)>0$.

By Protter [18], p.27, we can write $X=\tilde{X}+\hat{X}$ where $\tilde{X}_t:=(\tilde{\xi}_t,\tilde{\eta}_t)$ is a Lévy process with jumps contained in $\Lambda\setminus\Gamma$ and $\hat{X}_t:=(\hat{\xi}_t,\hat{\eta}_t)$ is a compound Poisson process independent of $\tilde{X}$, with jumps in Γ and parameter $\lambda:=\Pi_X(\Gamma)<\infty$. So we can write $\hat{X}_t=\sum_{i=1}^{N_t}C_i$ where N is a Poisson process with parameter λ and $(C_i)_{i\ge1}:=(C_i',C_i'')_{i\ge1}$ is an independent identically distributed sequence of two dimensional random vectors, independent of N, with $C_i\in\Gamma$. Let M be a Poisson process independent of N, C_i and $\tilde{X}$, with parameter $r\lambda$ for some $r\ge1$. Define the Lévy process Y by $Y_t:=\tilde{X}_t+\sum_{i=1}^{M_t}C_i$. We show the induced probability measures of X^T and Y^T on $\big(D([0,T]\to\mathbb{R}),\mathscr{B}^{[0,T]}\big)$ are mutually absolutely continuous. Since $\tilde{X}$ is independent of both compound Poisson processes, this is equivalent to showing the induced probability measures of $\sum_{i=1}^{N_t}C_i$ and $\sum_{i=1}^{M_t}C_i$ are mutually absolutely continuous. Let $A\in\mathscr{B}^{[0,T]}$ and note that

$$P\left(\left(\sum_{i=1}^{N_t}C_i\right)_{0\le t\le T}\in A\right)=\sum_{n=0}^{\infty}P\left(\left(\sum_{i=1}^{N_t}C_i\right)_{0\le t\le T}\in A\middle|N_T=n\right)P\left(N_T=n\right). \quad (4.17)$$

Since N is a Poisson process, $P(N_t=n)>0$ for all $n\in\mathbb{N}$. Thus the left hand side of (4.17) is zero if and only if $P\left(\left(\sum_{i=1}^{N_t}C_i\right)_{0\le t\le T}\in A\middle|N_T=n\right)=0$ for all $n\in\mathbb{N}$.

For any Poisson processes, regardless of parameter, Kallenburg [8], p.179 shows that once we condition on the event that n jumps have occurred in time $(0,T]$, then the jump times are uniformly distributed over $(0,T]$. This implies that

$$P\left(\left(\sum_{i=1}^{N_t}C_i\right)_{0\le t\le T}\in A\middle|N_T=n\right)=P\left(\left(\sum_{i=1}^{M_t}C_i\right)_{0\le t\le T}\in A\middle|M_T=n\right).$$

Thus $P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\le t\le T}\in A\right)=0$ if and only if $P\left(\left(\sum_{i=1}^{M_t} C_i\right)_{0\le t\le T}\in A\right)=0$, which proves that the two measures are mutually absolutely continuous, as required.

Recall that $Y_t =: (\tau_t,\nu_t) = \tilde{X}_t + \sum_{i=1}^{M_t} C_i$ where $\tilde{X} := (\tilde{\xi},\tilde{\eta})$ and $C_i := (C_i', C_i'') \in \Gamma$. Thus $\nu_t = \tilde{\eta}_t + \sum_{i=1}^{M_t} C_i''$ which implies that $tE(\nu_1) = tE(\tilde{\eta}_1) + r\lambda t E(C_i'')$ where $E(\tilde{\eta}_1) > E(\eta_1) \ge 0$. Choosing $r = E(\tilde{\eta}_1)/|\lambda E(C_i'')|$ gives $E(\nu_1)=0$ as required.

Case 3: Suppose $\sigma_\eta = 0$, $\Pi_\eta((-\infty,0))=0$, and $d_\eta<0$, where we allow the possibility that $d_\eta=-\infty$. If $\Pi_\eta((0,\infty))=0$ then $\eta_t = d_\eta t$ is deterministic, and this possibility has been excluded. So $\Pi_\eta((0,\infty))>0$, and we can assume X has jumps contained in Λ where we define the set $\Lambda := \{(x,y)\in\mathbb{R}^2 : -a\le x\le a, 0<y\le a\}$. For any $0<b<a$ define the set $\Gamma^{(b)}\subset\Lambda$ by $\Gamma^{(b)} := \{(x,y)\in\mathbb{R}^2 : -a\le x\le a, b\le y\le a\}$.

We can write $X = \tilde{X}^{(b)} + \hat{X}^{(b)}$ where $\tilde{X}^{(b)} := (\tilde{\xi}_t^{(b)}, \tilde{\eta}_t^{(b)})$ is a Lévy process with jumps contained in $\Lambda\setminus\Gamma^{(b)}$ and $\hat{X}^{(b)} := (\hat{\xi}_t^{(b)}, \hat{\eta}_t^{(b)})$ is a compound Poisson process independent of $\tilde{X}^{(b)}$, with jumps in $\Gamma^{(b)}$ and parameter $\lambda^{(b)} := \Pi_X(\Gamma^{(b)})<\infty$.

If $d_\eta\in(-\infty,0)$ then $E\left(\tilde{\eta}_t^{(b)}\right) = d_\eta t + t\int_{(0,b)} x\Pi_\eta(\mathrm{d}x)$. Since $\lim_{b\downarrow 0}\int_{(0,b)} x\Pi_\eta(\mathrm{d}x)=0$, there exists $b>0$ such that $E\left(\tilde{\eta}_t^{(b)}\right)<0$. If $d_\eta=-\infty$ then $\int_{(0,1)} x\Pi_\eta(\mathrm{d}x)=\infty$. Note that $E(\eta_1) = E\left(\tilde{\eta}_1^{(b)}\right) + E\left(\hat{\eta}_1^{(b)}\right)\in(0,\infty)$ since jumps are bounded, whilst

$$\lim_{b\downarrow 0} E\left(\hat{\eta}_t^{(b)}\right) = \lim_{b\downarrow 0}\int_{(b,a)} x\Pi_\eta(\mathrm{d}x) = \infty.$$

Hence there again exists $b>0$ such that $E\left(\tilde{\eta}_t^{(b)}\right)<0$.

From now on we assume $b>0$ is small enough such that $E\left(\tilde{\eta}_t^{(b)}\right)<0$. Since a Lévy measure is σ-finite and $\Pi_\eta((0,\infty))>0$ we can also assume $\Pi_X(\Gamma^{(b)})>0$. Thus we drop the ${}^{(b)}$ from our labelling. We can write $\hat{X}_t = \sum_{i=1}^{N_t} C_i$ where N is a Poisson process with parameter λ and $(C_i)_{i\ge1} := (C_i', C_i'')_{i\ge1}$ is an independent identically distributed sequence of two dimensional random vectors, independent of N, with $C_i\in\Gamma$. Let M be a Poisson process independent of N, C_i and $\tilde{X}$, with parameter $r\lambda$ for some $r>0$. Define the Lévy process Y by $Y_t := \tilde{X}_t + \sum_{i=1}^{M_t} C_i$. Then the induced probability measures of X^T and Y^T are mutually absolutely continuous by the same proof as used in Case 2. If $Y =: (\tau,\nu)$ then $\nu_t = \tilde{\eta}_t + \sum_{i=1}^{M_t} C_i''$ with $C_i''\in[b,a]$. Since $E(\tilde{\eta}_1)<0$ for our choice of $0<b<a$, choosing $r = |E(\tilde{\eta}_1)|/\lambda E(C_i'')$ gives the result. □

Theorem 2.3. We first reduce to the case that $X=(\xi,\eta)$ has bounded jumps. Take a general (ξ,η), let $a>0$ and define

$$\Lambda := \{(x,y)\in\mathbb{R}^2 : -a\le x\le a, -a\le y\le a\}.$$

We can write $X = \tilde{X} + \hat{X}$ where $\tilde{X}_t := (\tilde{\xi}_t,\tilde{\eta}_t)$ is a Lévy process with jumps contained in Λ and $\hat{X}_t := (\hat{\xi}_t,\hat{\eta}_t)$ is a compound Poisson process, independent of $\tilde{X}$, with jumps in

$\mathbb{R}^2 \setminus \Lambda$, and parameter $\lambda := \Pi_X(\mathbb{R}^2 \setminus \Lambda) < \infty$. Note that

$$\hat{X}_t := \sum_{0 \le s \le t} \Delta X_s 1_{\mathbb{R}^2 \setminus \Lambda}(\Delta X_s)$$

and by Poisson properties, $P(\hat{X}_t = 0) > 0$ for any $t \ge 0$.

Suppose that $P\left(\int_0^T e^{-\tilde{\xi}_{s-}} \mathrm{d}\tilde{\eta}_s < 0\right) > 0$. Then $P(Z_T < 0) > 0$, because

$$\begin{aligned} P\left(\int_0^T e^{-\xi_{s-}} \mathrm{d}\eta_s < 0\right) &\ge P\left(\int_0^T e^{-\xi_{s-}} \mathrm{d}\eta_s < 0 \,\Big|\, \hat{X}_T = 0\right) P\left(\hat{X}_T = 0\right) \\ &= P\left(\int_0^T e^{-\tilde{\xi}_{s-}} \mathrm{d}\tilde{\eta}_s < 0 \,\Big|\, \hat{X}_T = 0\right) P\left(\hat{X}_T = 0\right) \\ &= P\left(\int_0^T e^{-\tilde{\xi}_{s-}} \mathrm{d}\tilde{\eta}_s < 0\right) P\left(\hat{X}_T = 0\right) \\ &> 0. \end{aligned}$$

Further, note that η is not a subordinator iff we can choose $a > 0$ such that $\tilde{\eta}$ is not a subordinator. If $\sigma_\eta^2 > 0$ or $d_\eta < 0$ then any $a > 0$ suffices. If $\Pi_\eta((-\infty, 0)) > 0$ then we can choose $a > 0$ large enough such that $\Pi_\eta((-a, 0)) > 0$. The converse is obvious. Thus the theorem is proved if we can prove it for the case in which the jumps are bounded. From now on assume that the jumps of $X = (\xi, \eta)$ are contained in the set Λ defined above. Note that this implies that $E(\eta_1)$ is finite.

If η is pure deterministic drift, then $\eta_t = d_\eta t$ where $d_\eta < 0$, since η is not a subordinator. In this case the theorem is trivial, since Z is strictly decreasing. Thus, assume that η is not deterministic drift. We first prove the theorem in the case that $-c := E(\eta_1) < 0$. Note that

$$\begin{aligned} P\left(Z_T < 0\right) &= P\left(\int_0^T e^{-\xi_{s-}} \mathrm{d}(\eta_s + cs) - \int_0^T e^{-\xi_{s-}} \mathrm{d}(cs) < 0\right) \\ &\ge P\left(\int_0^T e^{-\xi_{s-}} \mathrm{d}(\eta_s + cs) < 0\right) \\ &> 0. \end{aligned}$$

The final inequality follows by Lemma 4.1, which implies that $\int_0^T e^{-\xi_{s-}} \mathrm{d}(\eta_s + cs)$ is a martingale, so $E\left(\int_0^T e^{-\xi_{s-}} \mathrm{d}(\eta_s + cs)\right) = 0$. Note that $\int_0^T e^{-\xi_{s-}} \mathrm{d}(\eta_s + cs)$ is not identically zero due to our assumption that η is not deterministic drift.

Now we assume that $c := E(\eta_1) \ge 0$. Lemma 4.3 ensures there exists $Y := (\tau, \nu)$ with bounded jumps, adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, such that X^T and Y^T are mutually absolutely continuous for all $T > 0$, and $E(\nu_1) = 0$. If we let $W_t := \int_0^t e^{-\tau_{s-}} \mathrm{d}\nu_s$ then Lemma 4.1 ensures that W^T is a mean-zero martingale. We prove that W_T is not identically zero

Firstly if ν is deterministic drift then W is either strictly increasing, or strictly decreasing, hence W_T is not identically zero. If ν is not deterministic drift then the quadratic variation $[\nu,\nu]$ is an increasing process. Hence

$$\left[\int_0^\bullet e^{-\tau_{s-}}\mathrm{d}\nu_s, \int_0^\bullet e^{-\tau_{s-}}\mathrm{d}\nu_s\right]_T = \left(\int_0^T e^{-2\tau_{s-}}\mathrm{d}[\nu,\nu]_s\right) > 0.$$

If W_T is identically zero then W_t must be identically zero for all $t \leq T$, since W^T is a martingale. Thus $[W,W]_T = 0$, which gives a contradiction.

Since W is not identically zero, and $E(W_T) = 0$, we conclude $P(W_T < 0) > 0$. However, Lemma 4.2 ensures that the induced probability measures of Z^T and W^T are mutually absolutely continuous. Hence $P(Z_T < 0) > 0$. □

Theorem 2.1 follows from Theorems 3.2 and 3.4. So we now prove these theorems.

Theorem 3.2. Property 1 is immediate from the definition while Property 2 follows from the fact that V_t is increasing in z for all $t \geq 0$. Let W be the process such $e^{-\xi_t} = \epsilon(W)_t$. Then for any $u \in \mathbb{R}$,

$$\begin{aligned}
V_t &= e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} d\eta_s\right) \\
&= e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} d(\eta_s - uW_s) + u\int_0^t e^{-\xi_{s-}} dW_s\right) \\
&= e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} d(\eta_s - uW_s) + u(e^{-\xi_t} - 1)\right) \\
&= u + e^{\xi_t}\left(z - u + \int_0^t e^{-\xi_{s-}} d(\eta_s - uW_s)\right).
\end{aligned}$$

Now if $\eta - zW$ is a subordinator then $\int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s) \geq 0$ so $\delta(z) = z$. By Theorem 2.3 if $\eta - zW$ is not a subordinator then for some t and some $\epsilon > 0$,

$$P\left(\int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s) < -\epsilon\right) > 0$$

and so, with $V_0 = z + \epsilon$ and $u = z$,

$$\begin{aligned}
& P\left(\inf_{t\geq 0} V_t < z \middle| V_0 = z + \epsilon\right) \\
= \quad & P\left(\inf_{t\geq 0}\left\{z + e^{\xi_t}\left(\epsilon + \int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s)\right)\right\} < z\right) \\
> \quad & 0,
\end{aligned}$$

which implies that $\delta(z) \leq \delta(z+\epsilon) < z$ and establishes Property 3.

Property 3 implies Property 4 if $\eta-\delta(z)W$ is a subordinator. So suppose that $\eta-\delta(z)W$ is not a subordinator. Then from the argument above we know that for some $\epsilon > 0$, $\delta(\delta(z)+\epsilon) < \delta(z)$. Let $T_u = \inf\{t > 0 : V_t \le u\}$. By definition of δ we have that $P(T_{\delta(u)+\epsilon} < \infty) > 0$. By the strong Markov property of V_t, if $u < z$,

$$\begin{aligned} & P\left(\inf_{t\ge 0} V_t < \delta(u) \middle| V_0 = z\right) \\ = \quad & P\left(\inf_{t\ge 0} V_{t+T_{\delta(u)+\epsilon}} < \delta(u) \middle| V_0 = z\right) \\ = \quad & P\left(\inf_{t\ge 0} V_{t+T_{\delta(u)+\epsilon}} < \delta(u) \middle| T_{\delta(u)+\epsilon} < \infty, V_0 = z\right) P\left(T_{\delta(u)+\epsilon} < \infty\right) \\ \ge \quad & P\left(\inf_{t\ge 0} V_t < \delta(u) \middle| V_0 = \delta(u)+\epsilon\right) P\left(T_{\delta(u)+\epsilon} < \infty\right) \\ > \quad & 0. \end{aligned}$$

This contradiction proves Property 4. □

Proposition 3.1. This proof is similar to a proof in Bertoin et al. [3]. We reference this paper for one of the tedious calculations. Protter [18], p.84, proves the following formula, and shows that it defines a finite valued semimartingale:

$$\epsilon(W)_t = e^{W_t - \frac{1}{2}[W,W]_t^c} \prod_{0<s\le t} (1+\Delta W_s)e^{-\Delta W_s},$$

where $[W,W]^c$ denotes the path-by-path continuous part of $[W,W]$. Thus

$$-\xi_t = \ln \epsilon(W)_t = W_t - \frac{1}{2}[W,W]_t^c + \sum_{0<s\le t} \left(\ln(1+\Delta W_s) - \Delta W_s\right). \tag{4.18}$$

So $\Delta\xi_t = -\ln(1+\Delta W_t)$ whenever $\Delta W_t \in (-1,\infty)$, and correspondingly,

$$\Delta W_t = e^{-\Delta\xi_t} - 1. \tag{4.19}$$

This proves the statements concerning the Lévy measures $\Pi_{\xi,W}$ and Π_W.

It is easy to show that $[W,W]_t^c = \sigma_W^2 t$ whenever W is a Lévy process. It also follows easily from the definition of the random measure $N_{W,t}(\cdot,\mathrm{d}x)$ that

$$\sum_{0<s\le t} \left(\ln(1+\Delta W_s) - \Delta W_s\right) 1_\Lambda(\Delta W_s) = \int_\Lambda \left(\ln(1+x) - x\right) N_{W,t}(\cdot,\mathrm{d}x)$$

whenever 0 is not contained within the closure of Λ. Since Protter [18], p85, shows that the series $\sum_{0<s\le t} \left(\ln(1+\Delta W_s) - \Delta W_s\right)$ converges a.s, it follows that the equality holds for any Λ. Hence (4.18) becomes

$$-\xi_t = W_t - \frac{1}{2}\sigma_W^2 t + \int_{(-1,\infty)} \left(\ln(1+x) - x\right) N_{W,t}(\cdot,\mathrm{d}x). \tag{4.20}$$

The Brownian motion component of a Lévy process is independent of the jumps and drift. Thus for equality to hold in the above equation, we must have $B_W = -B_\xi$, which proves (3.14).

The proof of (3.15) closely follows the method of proving Theorem 2.2 (iv) in [3], and we do not include it. $\square$

Theorem 3.4. The Lévy process $S^{(u)} := \eta - uW$ is a subordinator if and only if the following three conditions hold: $\sigma^2_{S^{(u)}} = 0$, $\Pi_{S^{(u)}}((-\infty, 0)) = 0$, and $d_{S^{(u)}} \geq 0$ where $d_{S^{(u)}} := E\left(S_1^{(u)} - \int_{(0,\infty)} zN_{S^{(u)},1}(\cdot, \mathrm{d}z)\right)$.

Note that $\sigma^2_{S^{(u)}} = 0$ is equivalent to $B_\eta - uB_W = 0$, which is equivalent to $B_\eta = -uB_\xi$ by Proposition 3.1, which establishes (3.16).

We show that $S^{(u)}$ has no negative jumps if and only at least one of the dot point conditions of the theorem hold. Using (4.19) we see that $\Delta S_t^{(u)} = \Delta\eta_t - u\left(e^{-\Delta\xi_t} - 1\right)$. If $u \geq 0$ then $\Delta S_t^{(u)} < 0$ requires $(\Delta\xi_t, \Delta\eta_t)$ be contained within A_2, A_3, or A_4. Every $(\Delta\xi_t, \Delta\eta_t) \in A_3$ produces a $\Delta S_t^{(u)} < 0$. Recall that the value θ_2 is the supremum of all the values of $u \geq 0$ at which there can be a negative jump $\Delta S_t^{(u)}$ with $(\Delta\xi, \Delta\eta) \in A_2$. Note that at $u = \theta_2$ such a jump is not possible. The obvious symmetric statement holds for θ_4. Hence, if $u \geq 0$ then $S^{(u)}$ can have no negative jumps if and only if $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \leq \theta_4$ and $u \in [\theta_2, \theta_4]$.

If $u \leq 0$ then $\Delta S_t^{(u)} < 0$ requires $(\Delta\xi_t, \Delta\eta_t)$ be contained within A_1, A_2, or A_3. Every $(\Delta\xi_t, \Delta\eta_t) \in A_2$ produces a $\Delta S_t^{(u)} < 0$. Recall that the value θ_1 is the supremum of all the values of $u \leq 0$ at which there can be a negative jump $\Delta S_t^{(u)}$ with $(\Delta\xi, \Delta\eta) \in A_1$, and at $u = \theta_1$ such a jump is not possible. The obvious symmetric statement holds for θ_3. Hence, if $u \leq 0$ then $S^{(u)}$ can have no negative jumps if and only if $\Pi_{\xi,\eta}(A_2) = 0$, $\theta_1 \leq \theta_3$ and $u \in [\theta_1, \theta_3]$.

Finally, if $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ then $\theta_3 = \theta_2 = 0$ and so both of the above are satisfied when $u \in [\theta_1, \theta_4]$.

We now show that when the above two conditions hold, $d_{S^{(u)}} \geq 0$ is equivalent to (2.10.) We first use (1.6) to convert (3.15) into a relationship between the constants γ_ξ and γ_W, from the individual characteristic triplets of ξ and W. It becomes

$$\gamma_\xi + \gamma_W = \frac{1}{2}\sigma_\xi^2 + \int_{\mathbb{R}} \left(x1_{(-1,1)}(x) + (e^{-x} - 1)1_{(-\ln 2,\infty)}(x)\right) \Pi_\xi(\mathrm{d}x). \tag{4.21}$$

Note that for any Borel set Λ

$$\begin{aligned}\int_\Lambda zN_{\eta-uW,1}(\cdot,\mathrm{d}z) &= \int_{\{x+y\in\Lambda\}}(x+y)N_{-uW,\eta,1}(\cdot,\mathrm{d}(x,y))\\ &= \int_{\{y-ux\in\Lambda\}}(y-ux)N_{W,\eta,1}(\cdot,\mathrm{d}(x,y))\\ &= \int_{\{y-u(e^{-x}-1)\in\Lambda\}}\left(y-u(e^{-x}-1)\right)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y)).\end{aligned}$$

The expected value of each of the Brownian motion components of η and W is zero, as is the expected value of the compensated small jump processes of η and W. Thus

$$\begin{aligned}&d_{S^{(u)}}\\ =\;& E\left(\eta_1-uW_1-\int_{(0,\infty)}zN_{\eta_1-uW_1}(\cdot,\mathrm{d}z)\right)\\ =\;& \gamma_\eta-u\gamma_W+E\Big(\int_{|y|\ge1}yN_{\eta,1}(\cdot,\mathrm{d}y)-u\int_{|x|\ge1}xN_{W,1}(\cdot,\mathrm{d}x)\\ &\quad-\int_{(0,\infty)}zN_{\eta_1-uW_1}(\cdot,\mathrm{d}z)\Big)\\ =\;& \gamma_\eta-u\gamma_W+E\Big(\int_{|y|\ge1}yN_{\eta,1}(\cdot,\mathrm{d}y)-u\int_{(-\infty,-\ln 2)}\left(e^{-x}-1\right)N_{\xi,1}(\cdot,\mathrm{d}x)\\ &\quad-\int_{\{y-u(e^{-x}-1)>0\}}\left(y-u(e^{-x}-1)\right)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\Big)\\ =\;& \gamma_\eta+u\gamma_\xi-\frac12u\sigma_\xi^2+E\Big(\int_{\mathbb{R}^2}\Big(y1_{|y|\ge1}-ux1_{|x|<1}-u(e^{-x}-1)\\ &\quad-\left(y-u(e^{-x}-1)\right)1_{\{y-u(e^{-x}-1)>0\}}\Big)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\Big)\\ =\;& \gamma_\eta+u\gamma_\xi-\frac12u\sigma_\xi^2\\ &\quad-E\left(\int_{\{y-u(e^{-x}-1)>0\}\cap\{(-1,1)\times(-1,1)\}}(ux+y)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\right)\\ =\;& \tilde\gamma_\eta+u\tilde\gamma_\xi-\frac12u\sigma_\xi^2\\ &\quad-E\left(\int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}}(ux+y)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\right),\end{aligned}$$

where the third equality follows using (4.21), the fourth equality follows since $S^{(u)}$ has no negative jumps, so $N_{\xi,\eta,1}(\{y-u(e^{-x}-1)\le0\})=0$, and the final equality follows by (1.6). Thus we are done if we can exchange integration and expectation in the above expression. Now if $f(x,y)$ is a non-negative measurable function and Λ is a Borel set in $\mathbb{R}^2$ then the monotone convergence theorem implies that

$$E\left(\int_\Lambda f(x,y)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\right)=\int_\Lambda f(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)).$$

For general $f(x,y)$, if $\int_\Lambda f^+(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y))$ or $\int_\Lambda f^-(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y))$ is finite, then the following is well-defined;

$$\begin{aligned}
& E\left(\int_\Lambda f(x,y)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\right) \\
= & \int_\Lambda f^+(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)) - \int_\Lambda f^-(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\
= & \int_\Lambda f(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)).
\end{aligned}$$

However, using the fact that $0 < e^{-x} - 1 + x < x^2$ whenever $|x| < 1$, we have

$$\begin{aligned}
& \int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}} (ux+y)^-\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\
= & \int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}} -(ux+y)1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\
\le & \int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}} \big(y - u(e^{-x}-1) - (ux+y)\big)\, 1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\
= & \int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}} -u(e^{-x}-1+x)1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\
\le & \int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}} |u|x^2 1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\
\le & |u|\int_{\mathbb{R}} \min\left\{1,x^2\right\}\Pi_\xi(\mathrm{d}x),
\end{aligned}$$

which is finite since Π_ξ is a Lévy measure. $\square$

Theorem 2.1. Clearly $\psi(z) = 0$ if and only if $\delta(z) \ge 0$. By Theorem 3.2, this is equivalent to the condition that there exists $0 \le u \le z$ such that $\delta(u) = u$. Combining this fact with Theorem 3.4 proves Theorem 2.1. $\square$

Theorem 2.4. Define

$$U_t := e^{\xi_t}(Z_\infty - Z_t) = e^{\xi_t}\int_{t+}^\infty e^{-\xi_{s-}}\mathrm{d}\eta_s.$$

Note that since we are integrating over (t,∞) there are no predictability problems moving e^{ξ_t} under the integral sign, as there would have been if we were integrating over $[t,\infty)$. Thus $U_t = \int_{t+}^\infty e^{-(\xi_{s-}-\xi_t)}\mathrm{d}\eta_s$, from which it follows, from Lévy properties, that U_t is independent of $\mathscr{F}_t$ and that U_{T_z} conditioned on $T_z < \infty$ is independent of $\mathscr{F}_{T_z}$.

Since (ξ,η) is a Lévy process we know that for any $u > 0$ and $t > 0$

$$(\hat{\xi}_{u-},\hat{\eta}_u) := (\xi_{(t+u)-} - \xi_t\ ,\ \eta_{t+u} - \eta_t) =_D (\xi_{u-},\eta_u). \tag{4.22}$$

Thus

$$\begin{aligned}
U_t &= \int_{t+}^{\infty} e^{-(\xi_{s-}-\xi_t)}\mathrm{d}\eta_s = \int_{0+}^{\infty} e^{-(\xi_{(t+u)-}-\xi_t)}\mathrm{d}\eta_{t+u} \\
&= \int_{0+}^{\infty} e^{-(\xi_{(t+u)-}-\xi_t)}\mathrm{d}(\eta_{t+u}-\eta_t) = \int_{0+}^{\infty} e^{-\hat{\xi}_{u-}}\mathrm{d}\hat{\eta}_u \\
&=_D \int_{0+}^{\infty} e^{-\xi_{u-}}\mathrm{d}\eta_u \quad \text{(by (4.22))} = Z_\infty \quad \text{(since } \Delta\eta_0 = 0).
\end{aligned}$$

In particular, for any Borel set A,

$$P\left(U_{T_z} \in A \middle| T_z < \infty\right) = P(Z_\infty \in A). \tag{4.23}$$

Next note that if $\omega \in \{T_z < \infty\}$ then by definition of U,

$$\begin{aligned}
z + Z_\infty &= z + Z_{T_z} + e^{-\xi_{T_z}} U_{T_z} \\
&= e^{-\xi_{T_z}}\left(e^{\xi_{T_z}}(z + Z_{T_z}) + U_{T_z}\right) \\
&= e^{-\xi_{T_z}}(V_{T_z} + U_{T_z}).
\end{aligned}$$

This implies that

$$P(T_z < \infty, z + Z_\infty < 0) = P(T_z < \infty, V_{T_z} + U_{T_z} < 0). \tag{4.24}$$

Finally note that $(Z_\infty < -z) \subset (T < \infty)$ since the convergence from Z_t to Z_∞ is a.s. Thus

$$\begin{aligned}
P(z + Z_\infty < 0) &= P(T_z < \infty, z + Z_\infty < 0) \\
&= P(T_z < \infty, V_{T_z} + U_{T_z} < 0) \quad \text{(by (4.24))} \\
&= E\left(P(T_z < \infty, V_{T_z} + U_{T_z} < 0 \middle| \mathscr{F}_{T_z})\right) \\
&= \int_{T_z < \infty} P\left(V_{T_z} + U_{T_z} < 0 \middle| \mathscr{F}_{T_z}\right)(\omega) P(\mathrm{d}\omega).
\end{aligned}$$

But if $T_z(\omega) < \infty$ then

$$\begin{aligned}
P\left(V_{T_z} + U_{T_z} < 0 \middle| \mathscr{F}_{T_z}\right)(\omega) &= P\left(V_{T_z}(\omega) + U_{T_z} < 0 \middle| \mathscr{F}_{T_z}\right)(\omega) \\
&= P\left(U_{T_z} < -V_{T_z}(\omega) \middle| T_z < \infty\right) \\
&= P\left(Z_\infty < -V_{T_z}(\omega)\right) \quad \text{(by (4.23))}.
\end{aligned}$$

The second last equality follows since U_{T_z} conditioned on $T_z < \infty$ is independent of $\mathscr{F}_{T_z}$. Thus we obtain the required formula from

$$
\begin{aligned}
G(-z) &= \int_{T_z<\infty} G(-V_{T_z})(\omega) P(\mathrm{d}\omega) \\
&= E\left(G(-V_{T_z}) 1_{T_z<\infty}\right) \\
&= E\left(G(-V_{T_z}) 1_{T_z<\infty} \middle| T_z < \infty\right) P(T_z < \infty) \\
&\qquad + E\left(G(-V_{T_z}) 1_{T_z<\infty} \middle| T_z = \infty\right) P(T_z = \infty) \\
&= E\left(G(-V_{T_z}) \middle| T_z < \infty\right) P(T_z < \infty).
\end{aligned}
$$

□

Acknowledgements. We are grateful to Professor Ross Maller for several close readings of the paper and constructive comments which helped us to substantially improve the readability.

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